DIVISION OF THE HUMANITIES AND SOCIAL SCIENCES

CALIFORNIA INSTITUTE OF TECHNOLOGY

PASADENA, CALIFORNIA 91125

CLASSIFICATION OF VOTING GAMES ON MANIFOLDS

Norman Schofield

SOCIAL SCIENCE WORKING PAPER 488

July 1983

CLASSIFICATION OF VOTING GAMES ON MANIFOLDS*

Norman Schofield

ABSTRACT

A voting game σ is classified by two integers $v^*(\sigma), w^*(\sigma), (v^*(\sigma) < w^*(\sigma))$. In dimension $< w^*(\sigma)$ the existence of the σ-core is <u>structurally stable</u> (in the C^1-topology on smooth profiles); in dimension $> v^*(\sigma)$ the emptiness of the σ-core is structurally stable.

* Thanks are due to Gary Cox, of the University of Texas at Austin, for making available some of his unpublished work. The results presented here are much influenced by Cox's work.

CLASSIFICATION OF VOTING GAMES ON MANIFOLDS*

Norman Schofield

Introduction

The purpose of this note is to present a classification theorem for voting games. Any voting game, σ, is essentially classified by two integers, $v^*(\sigma)$ and $w^*(\sigma)$, which we shall call the stability and instability dimensions. Suppose that W is a policy space of dimension w; by that we mean that W is either a smooth w-dimensional manifold (perhaps with boundary) or is a convex subset of a topological vector space. In the latter case the dimension of W is the dimension of the affine manifold generated by W. In either case we shall simply call W a w-manifold. Let $N = \{1,\ldots,i,\ldots,n\}$ be the society. The voting game σ is characterised by a family $\mathcal{D}$ of subsets of N. Each member of $\mathcal{D}$ is called a winning coalition.

Suppose now that $u = (u_1,\ldots,u_i,\ldots,u_n):W \to \mathbb{R}^n$ is a smooth profile for N on the w-manifold W. A standard object of study is the core. Say a point x is dominated by a point y, and write $y\sigma(u)x$, if and only if there is a coalition M in $\mathcal{D}$ such that $u_i(y) > u_i(x)$ for all i in M. A point x in W belongs to the core, or global optima set, $GO(\sigma,u)$, of $\sigma(u)$ if and only if there is no point y in W such that $y\sigma(u)x$. We shall also say that a point x in W belongs to the global cycle set, $GC(\sigma,u)$, of $\sigma(u)$ if and only if there exists a finite subset $\{y_1,\ldots,y_r\}$ in W such that

$$x\sigma(u)y_1\ \sigma(u)\ \ldots\sigma(u)y_rx.$$

Since preferences are smooth it is possible to define two differential analogues of $GO(\sigma,u)$ and $GC(\sigma,u)$, which we shall call the critical (or infinitesimal) optima and cycle sets $IO(\sigma,u)$ and $IC(\sigma,u)$. Previous analyses (Schofield 1978,1980) have shown that $GO(\sigma,u)$ is a subset of $IO(\sigma,u)$ and $IC(\sigma,u)$ is a subset of $GC(\sigma,u)$. It has previously been shown (Walker,1977) that when W is compact, and $GC(\sigma,u)$ is empty, then $GO(\sigma,u)$ must be non empty. Moreover if W is a manifold of a certain topological kind (for example contractible) then if $IC(\sigma,u)$ is empty, $IO(\sigma,u)$ must be non-empty (Schofield 1983a).

Finally, if all preferences are convex, then $IO(\sigma,u)$ and $GO(\sigma,u)$ coincide. Thus knowledge on the existence or otherwise of $IC(\sigma,u)$ and $IO(\sigma,u)$ provides information on the existence of a core of the voting game (σ,u).

The classification we propose is in terms of the existence of the sets $IO(\sigma,u)$ and $IC(\sigma,u)$ as u varies across all smooth profiles.

First of all let $U_r(W)^N$ be the space of smooth profiles for N on W, endowed with the Whitney C^r-topology. In the case that W is a compact subset of $\mathbb{R}^w$ this topology can be given an easy interpretation. A profile v belongs to a neighbourhood $N_r(u,\delta)$ of a profile u if and only if, for all $x \in W$ for all $i \in N$, and for all $k = 0,\ldots,r$,

$$||du_i^k(x) - dv_i^k(x)|| < \delta.$$

Here d^ku_i is the k^{th} differential of $u_i:W \to \mathbb{R}$ (where we use the notation that the zeroth differential $du_i^0 = u_i$).

The C^r-topology can be defined when W is a smooth manifold, even for W not compact (see Hirsch 1976).

Note that the C^r-topology is finer than the C^s-topology whenever $r > s$. That is to say that a set which is open in $U_s(W)^N$ must be open in $U_r(W)^N$, but the converse need not be true. In particular an open set in the C^1-topology $U_1(W)^N$ need not be open in $U_0(W)^N$.

Suppose now that W is a w-manifold and σ a voting game. Define the <u>stable subspace</u> of profiles to be

$$K_1(\sigma,W) = \{u \in U_1(W)^N : IO(\sigma,u) \neq \Phi \text{ and } IC(\sigma,u) = \Phi\}$$

and the <u>unstable subspace</u> of profiles to be

$$\mathcal{L}_1(\sigma,W) = \{u \in U_1(W)^N : IO(\sigma,u) = \Phi \text{ and } IC(\sigma,u) \neq \Phi\}$$

(Here Φ is the empty set in W)

Note that we use the C^1-topology. For convenience in notation we shall simply write K for K_1 and $U(W)^N$ for $U_1(W)^N$ etc.

<u>Definition</u>

i) The <u>stable dimension</u> $v^*(\sigma)$ is the <u>smallest</u> integer such that for any w-manifold, W, if $w > v^*(\sigma)$ then $\mathcal{L}(\sigma,W)$ has non empty interior in $U(W)^N$

ii) The <u>unstable dimension</u> $w^*(\sigma)$ is the greatest integer such that, for any w-manifold, W, if $w < w^*(\sigma)$ then $K(\sigma,W)$ has non empty interior in $U(W)^N$.

In the following sections of the paper we show that the stable and unstable dimensions do exist, and illustrate how they may be computed. By this method we obtain a classification theorem for all voting games in the case of smooth profiles.

Consider the nature of the classification. From (ii) for example it must be possible to find a manifold W of dimension $w^*(\sigma)$ such that,

if u is a profile with $IO(\sigma,u) \neq \Phi$ then, in <u>any</u> neighbourhood V of u in $U(W)^N$, there exists a profile v such that $IO(\sigma,v) = \Phi$. We shall also say that the property "$IO(\sigma,u) \neq \Phi$" is <u>structurally unstable</u>, since it is destroyed by arbitrary small perturbation. On the other hand if W is any manifold of dimension $w \leq w^*(\sigma)-1$ then it is possible to find a profile u and a neighbourhood V of u in $U(W)^N$ such that $IO(\sigma,v) \neq \Phi$ for all v in V. Thus "$IO(\sigma,u) \neq \Phi$" is <u>structurally stable</u>

<u>The stability dimension</u>

We first of all give the definitions of the two critical sets $IO(\sigma,u)$ and $IC(\sigma,u)$, associated with a game σ and profile u.

For a coalition M, define the set $IO(M,u)$ in W by $x \in IO(M,u)$ iff there exists no $y \in W$ such that for all $i \in M$, $du_i(x)(y-x) > 0$. Define the critical optima set of $\sigma(u)$ by $IO(\sigma,u) = \bigcap_{M \in \mathcal{D}} IO(M,u)$.

Here $\mathcal{D}$ is the set of winning coalitions of σ.

At a point $x \in W$, define $\mathcal{D}(x) = \{M \in \mathcal{D} : x \notin IO(M,u)\}$.

For coalition $M \subset N$, let $p_M(x)$ be the convex hull of $\{du_i(x) : i \in M\}$.

Define the <u>directional core</u> at x to be

$$p_\sigma(x) = \bigcap_{\mathcal{D}(x)} p_M(x)$$

and define the critical cycle set of $\sigma(u)$ by

$$IC(\sigma,u) = \{x \in W : p_\sigma(x) = \Phi\}.$$

As mentioned previously, it has been shown (Schofield,1978,1980) that

$GO(\sigma,u) \subset IO(\sigma,u)$

and $IC(\sigma,u) \subset GC(\sigma,u)$.

We now define the stability dimension $v^*(\sigma)$.

For any family $\mathcal{D}$ of subsets of N, let $C(\mathcal{D}) = \bigcap_{M\in\mathcal{D}} M$ be the collegium of $\mathcal{D}$.
If σ is a voting game with winning coalitions $\mathcal{D}$ such that $C(\mathcal{D}) \neq \Phi$ then call σ collegial and define $v(\sigma) = \infty$.

If $C(\mathcal{D}) = \Phi$ then define the Nakamura (1978) number, $v(\sigma)$, of σ to be

$$v(\sigma) = \min\{|\mathcal{D}'| : \mathcal{D}' \subset \mathcal{D} \text{ and } C(\mathcal{D}') = \Phi\}.$$

For example suppose σ is proper: if A,B belong to $\mathcal{D}$ then $A \cap B \neq \Phi$. In this case if $\mathcal{D}' = \{A,B\}$ then $C(\mathcal{D}') \neq \Phi$ and so $v(\sigma) \geq 3$.

A q-majority game, σ_q, has winning coalitions $\mathcal{D}_q = \{M \subset N: |M| \geq q\}$.
It is easy to show that for a q-game, when $q < n$, then
$v(\sigma_q) = v(n,q)+2$, where $v(n,q)$ is the greatest integer which is strictly less than $q/n-q$.

Strict majority rule σ_m is the q-game given by:

$$(n,q) = \begin{cases}(2k,k+1) \\ (2k+1,k+1)\end{cases} \quad \text{if n is} \begin{cases}\text{even} \\ \text{odd}.\end{cases}$$

In the case that $(n,q) = (4,3)$ then $v(n,q) = 2$ and so $v(\sigma) = 4$.
For any other strict majority rule, σ_m, $v(\sigma_m) = 3$.

It has been shown (Schofield 1983b) that if W is a smooth manifold of dimension $w \leq v(\sigma)-2$ then, for any smooth profile on W, the cycle set $IC(\sigma,u)$ must be empty. Moreover if W is a compact convex set in $\mathbb{R}^W$, and $w \leq v(\sigma)-2$ then the optima set must be non empty (Schofield 1983c). Consequently it is possible to find a smooth manifold W of dimension $v(\sigma)-2$ such that, for all $u \in U(W)^N$, $IC(\sigma,u) = \Phi$ and $IO(\sigma,u) \neq \Phi$. Thus $\mathcal{L}(\sigma,W)$ is itself empty.

This suggests that $v(\sigma)-2$ is a candidate for the stability dimension $v^*(\sigma)$.

Strnad (1981,1982) has shown that if W is a convex manifold of dimension $v(\sigma)-1$ then it is possible to construct a smooth profile u on W such that $IO(\sigma,u) = \Phi$. Adapting Strnad's method somewhat, Schofield (1983d) showed, for any smooth manifold W of dimension at least $v(\sigma)-1$, that a smooth profile u on W could be constructed such that $IC(\sigma,u) \neq \Phi$ and $IO(\sigma,u) = \Phi$. Moreover this property was structurally stable, i.e. true for any profile in some neighbourhood of u in $U(W)^N$.
Finally note that if σ is collegial, then by definition $v(\sigma) = \infty$. In Schofield (1983b) it was shown that if σ was collegial, then for any $u \in U(W)^N$, $IC(\sigma,u) = \Phi$.
Thus we obtain:

Theorem 1 For any voting game, σ, it is the case that the stability dimension $v^*(\sigma)$ equals $v(\sigma)-2$.

One further minor point. At dimension $v^*(\sigma)+1$ it is possible to show that when $IC(\sigma,u)$ is non empty then it must belong to the critical pareto set $IO(N,u)$. In dimension $v^*(\sigma)+2$ and above this is not the case.

The instability dimension

Suppose that for some voting game, σ, it was the case that $w^*(\sigma) \leq v^*(\sigma)$. As we have observed, in dimension $v^*(\sigma)$ it is always the case that $IC(\sigma,u) = \Phi$. As a result it is possible to construct a profile u in this dimension such that the property "$IO(\sigma,u) \neq \Phi$" is structurally stable. Consequently it must be the case that $v^*(\sigma) < w^*(\sigma)$. If σ is a collegial game then $w^*(\sigma) = \infty$.

Suppose then that σ is a non-collegial game with $\mathcal{D}$ its winning coalitions.

For each subfamily $\mathcal{D}'$ of $\mathcal{D}$ such that $C(\mathcal{D}') = \Phi$ define

$w(\mathcal{D}') = \max \{|M| : M \in \mathcal{D}'\}$,

$w(\sigma) = \min \{ w(\mathcal{D}') : \mathcal{D}' \subset \mathcal{D}$ and $C(\mathcal{D}') = \Phi\}$.

For example in a q-game, σ_q (with $q < n$), it is possible to find a family $\mathcal{D}'$ of $(q+1)$ distinct coalitions, each with q members, such that $C(\mathcal{D}') = \Phi$. Thus $w(\sigma_q) \leq q$. For an arbitrary non-collegial game it certainly must be the case that $w(\sigma) \leq n-1$. For a collegial voting game define $w(\sigma) = \infty$.

Schofield (1980) showed that if σ was a non collegial voting game on a smooth w-manifold without boundary, and $w \geq w(\sigma)$ then $IO(\sigma,u)$ was <u>almost always</u> or generically empty.

To be more precise, a <u>residual</u> set R in $U(W)^N$ is a dense set formed by the countable intersection of open dense sets in $U(W)^N$. If W is compact then R will also be open. A generic property is one that is true for all profiles in a residual set. From standard results in singularity theory, the critical optima set for a coalition M in $\mathcal{D}$ is generically a geometric object of dimension at most $|M|-1$. In dimension $w(\sigma)$, a counting argument shows that the critical optima sets of the winning coalitions generically do not intersect.

Hence if W is a w-manifold without boundary, and if $w \geq w(\sigma)$, then for all u in a dense set $R(\sigma,W)$ in $U(W)^N$, it is the case that $IO(\sigma,u) = \Phi$. Suppose now that $u \in K(\sigma,W)$, and so $IO(\sigma,u) \neq \Phi$. Since $u \notin R(\sigma,W)$ and $R(\sigma,W)$ is dense, any neighbourhood V of u must intersect $R(\sigma,W)$. Consequently $K(\sigma,W)$ cannot have a non empty interior for a manifold, W, without boundary.

Thus we obtain:

<u>Theorem 2</u> For any voting game σ, it is the case that $w^*(\sigma) \leq w(\sigma)$.

Notice that $w(\sigma)$ is only an upper bound on the instability dimension $w^*(\sigma)$. In some cases an upper bound lower than $w(\sigma)$ can be obtained. For example, if σ_q is a q-game, let $w(n,q)$ be the greatest integer which is less than $q - \frac{n}{2} + 1$ (assuming that $q > \frac{n}{2}$). Cox (1983) effectively has shown that if W is a w-manifold, and $w \leq w(n,q)$, then $K(\sigma,W)$ has a non empty interior.

Thus $w^*(\sigma_q)$ must lie in the range $[w(n,q)+1,q]$.

For a strict majority rule, σ_m, the instability dimension $w^*(\sigma_m)$ can be precisely calculated. Using results of Plott (1967) and Matthews (1982), it has been shown (Schofield 1983e) that, for majority rule σ_m, $w^*(\sigma_m) = 2$ if the society is of odd size and $w^*(\sigma_m) = 3$ for a society of even size. With a society of odd size $w(2k+1,k+1) = 1$ and with an even size $w(2k,k+1) = 2$.

In general if W is a w-manifold without boundary, and $w \geq w^*(\sigma)$ then the stable profile subspace $K(\sigma,W)$ is nowhere dense. However if W has a boundary and $w = w^*(\sigma)$, then $K(\sigma,W)$ may have a non empty interior. However for any profile u in the interior of $K(\sigma,W)$ it must be the case that the (non empty) optima set, $IO(\sigma,u)$, belongs to the boundary of W.

If $w \geq w^*(\sigma)+1$, then, even when W has a boundary, $K(\sigma,W)$ must be nowhere dense. In this dimension range it has been shown that $\mathcal{L}(\sigma,W)$ not only has a non empty interior, but is itself dense. This result is obtained by showing that when $w > w^*(\sigma)$ then $IC(\sigma,u)$ is generically not only non empty, but is itself dense in W.

It is possible that for a non-collegial voting game, σ, the dimension range $[v^*(\sigma)+1, w^*(\sigma)-1]$ is non empty. In this case there may exist a manifold W such that both $K(\sigma,W)$ and $\mathcal{L}(\sigma,W)$ have non empty interior. It is therefore possible to find a profile u such that "$IO(\sigma,u) \neq \Phi$" is structurally stable, and a second profile v such that "$IC(\sigma,v) \neq \Phi$" is also structurally stable.

Consider for example strict majority rule σ_m. We know that $v^*(\sigma_m) = 1$, except for the case $(n,q) = (4,3)$.

If n is even then $w^*(\sigma_m) = 3$, and so the two dimensional situation is an intermediate case, in the sense that both cycles and optima may exist in a structurally stable way. If n is odd however $w^*(\sigma_m) = 2$, and so "structurally stable optima" cannot be found in the interior of W.

Insert Figure 1 about here

Figure 1 sums up these observations.

Discussion

Rubinstein (1978) has shown, without a dimension constraint, that the set of profiles, which admit a core, is nowhere dense in the space of profiles with the Kannai topology. Cox (1983) extended this by effectively showing that the set $K(\sigma,W)$ has empty interior in $U_o(W)^N$ (the space of profiles with the C^o-topology) for any non-collegial voting game, σ.

Suppose now that $\dim(W) \geq w^*(\sigma)+1$, for some manifold, W.

We know from the results referred to here that $K(\sigma,W)$ must have empty interior in $U_1(W)^N$. Since an open set in the C^o-topology is open in the C^1-topology,

this implies that $K(\sigma,W)$ has empty interior in $U_o(W)^N$.

But now consider the case $\dim(W) \leq v^*(\sigma)$, and suppose u is a convex preference profile. By Greenberg (1978) and Strnad (1981), $GO(\sigma,u) \neq \Phi$.

From Rubinstein's result, in any neighbourhood V of u in $U_o(W)^N$ there exists a profile v such that $GO(\sigma,v) = \Phi$. Of course such a profile must be non-convex. Although V is also open in $U_1(W)^N$, there will exist a neighbourhood V' of u in the <u>C^1-topology</u> such that $IO(\sigma,v') \neq \Phi$ for all v' in V'. Indeed it is plausible that for all v' in V' the core $GO(\sigma,v')$ is non empty. This suggests in fact that we may redefine the stable subspace to be $K_1'(\sigma,w) = \{u \in U_1(W)^N : GO(\sigma,u) \neq \Phi \text{ and } IC(\sigma,u) = \Phi\}$, and obtain the same results as those presented here for $K_1(\sigma,w)$. Note also that the unstable subspace $\mathcal{L}_1(\sigma,W)$ is contained in $\mathcal{L}_1'(\sigma,W) = \{u \in U_1(W)^N : GO(\sigma,u) = \Phi \text{ and } GC(\sigma,u) \neq \Phi\}$. Consequently the results presented here for $\mathcal{L}_1(\sigma,W)$ are true for $\mathcal{L}_1'(\sigma,W)$.

References

[1] Cox, G.W. (1983), Non Collegial Simple Games and the Nowhere Denseness of the Set of Preference Profiles having a Core. Mimeographed. University of Texas at Austin.

[2] Greenberg, J. (1979), Consistent Majority Rule over Compact Sets of Alternatives. Econometrica, 47, 627-636.

[3] Hirsch, M. W. (1976), Differential Topology. New York: Springer Verlag.

[4] Matthews, S. (1982), Local Simple Games in Public Choice Mechanisms, International Economic Review, 23, 623-645.

[5] Nakamura, K. (1978), The vetoers in a Simple Game with Ordinal Preference, Int. J. Game Theory, 8,55-61.

[6] Plott, C.R. (1967), A Notion of Equilibrium and its possibility Under Majority Rule, American Economic Review, 57, 787-806.

[7] Rubinstein, A. (1979), A Note about the "nowhere denseness" of societies having an equilibrium under majority rule, Econometrica,47,512-514.

[8] Schofield, N. (1978), Instability of Simple Dynamic Games, Review of Economic Studies, 45, 575-594.

[9] Schofield, N. (1980), Generic Properties of Simple Bergson Samuelson Welfare Functions, J. Math. Econ., 7, 175-192.

[10] Schofield, N. (1983a), Existence of Equilibrium on a Manifold, Forthcoming, Mathematics of Operations Research

[11] Schofield, N. (1983b), Equilibrium in Simple Dynamic Games, in Social Choice and Welfare (P. Pattanaik and M. Salles, Eds.) North Holland: New York.

[12] Schofield, N. (1983c), Social Equilibrium and Cycles on Compact Sets. Forthcoming, Journal of Economic Theory.

[13] Schofield, N. (1983d), The Geometry of Voting. Mimeographed: California Institute of Technology Working Paper #485.

[14] Schofield, N. (1983e), Generic Instability of Majority Rule. Forthcoming, Review of Economic Studies.

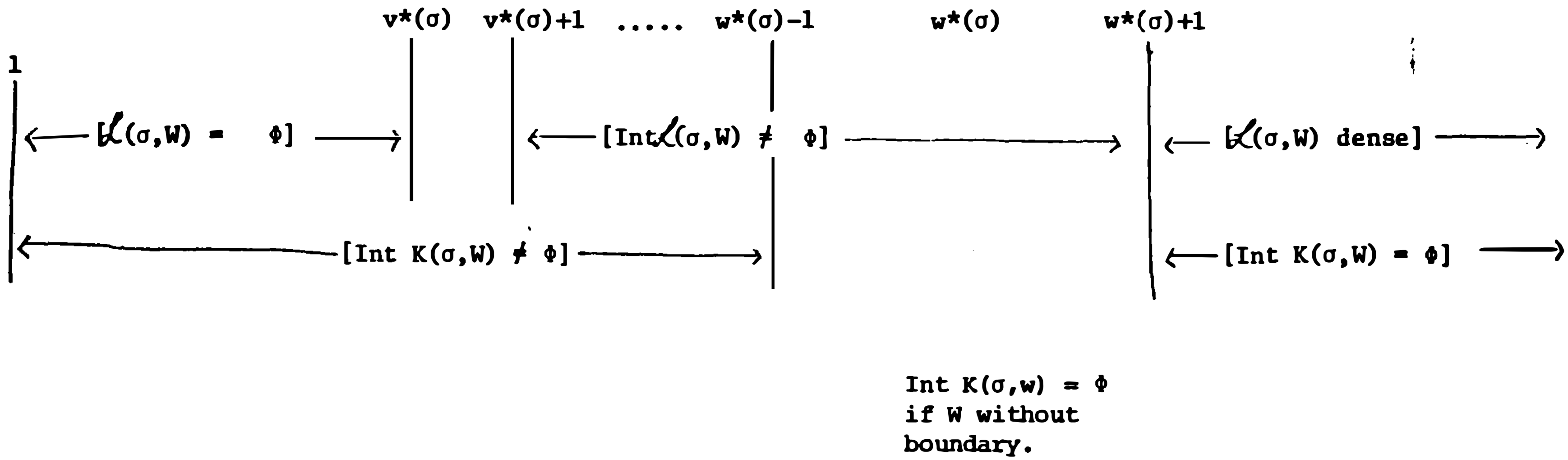
[15] Strnad, J. (1981), The Structure of Continuous Neutral Monotonic Social Functions. Mimeographed, University Of Southern California.

[16] Strnad, J. (1982), The Structure and Nash Implementation of

Neutral Monotonic Social Functions, Ph.D. Dissertation, Yale University, December 1982.

[17] Walker, M. (1977), On the Existence of Maximal Elements, Journal of Economic Theory, 16, 470-474.

Figure 1



Classification of a voting game, σ, in terms of the dimensions $v^*(\sigma)$, $w^*(\sigma)$ and the stable and unstable subspaces $K(\sigma,W)$, $\mathcal{L}(\sigma,W)$ in the space of smooth profiles $U_1(W)^N$.